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Minimal Triangulation of a Graph and Optimal Pivoting Order in a Sparse Matrix*

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This paper considers the problem of finding a minimal triangulation of an undirected graph $G = (V, E)$, where a triangulation is a set T such that every cycle in $\hat{G} = (V, E \cup T)$ has a chord. A triangulation T is *minimal* (minimum) if no triangulation F exists such that F is a proper subset of T ($|F| < |T|$), and an ordering α is *optimal* (optimum) if a minimal (minimum) triangulation is generated by α. A minimum triangulation (optimum ordering) is necessarily minimal (optimal), but the converse is not necessarity true. A necessary and sufficient condition for a triangulation to be minimal is presented. This leads to an algorithm for finding an optimal ordering α which produces a minimal set of "fill-in" when the process is viewed as triangular factorization of a sparse matrix.

1. INTRODUCTION

The problem of solving a large sparse system of linear equations arises in widespread applications, such as in analyses of electric and hydraulic networks, analyses of economic models, and structural analysis. Since, in most cases, coefficient matrices have a fixed sparseness structure in the entire

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course of computation, it is crucial to find a priori a good pivoting order for triangular factorization of such matrices.

After Parter introduced graph–theoretical concepts [1], certain investigations developed several heuristic ordering schemes which are practical in the sense that relatively simple operations are involved [2–5]. It is, however, well known that these heuristic algorithms may give poor orderings [6].

Recently, Rose introduced the concept of triangulated graphs [7–12] and established a solid foundation of the theory of optimal orderings, though he failed to give a correct algorithm of finding an optimal ordering [6]. It should be emphasized here that the word "optimal" should not be mistaken to mean "optimum" as in some engineering literature. In terms of a set, "optimal" refers to a set property, whereas "optimum" refers to the size of the set.

In this paper we further extend the theory due to Rose and give a complete characterization of *minimal* triangulations. Based on this result an algorithm is presented to find an *optimal* ordering or all optimal orderings if desired. In Sect. 2 the relationship between triangular factorization of a matrix and triangulation of the associated graph is discussed. In Sect. 3 a necessary and sufficient condition is given for a triangulation to be minimal. An example indicates the insufficiency of "local" information to find optimal orderings. Sect. 4 a complete characterization of optimal orderings is presented, which leads to an algorithm. Finally an example illustrates the algorithm.

2. Triangular Factorization of a Matrix and Triangulation of a Graph

A standard way of solving a system of linear equations is to use triangular factorization [13]. We discuss this process graph theoretically, following Rose [11]. Let

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

be the given system, where $\mathbf{A} = \{a_{ij}\}$ is a nonsingular $n \times n$ matrix. The coefficient matrix $\mathbf{A}$ is factored as

$$\mathbf{PAQ}^T = \mathbf{LDU}, \tag{2}$$

where $\mathbf{P}$, $\mathbf{Q}$ are row-permutation matrices, $\mathbf{D}$ is a diagonal matrix, and $\mathbf{L}, \mathbf{U}^T$ are unit lower triangular matrices. We restrict ourselves to the consideration of a class of matrices $\mathbf{A}$ satisfying the following assumptions.

(i) All the principal minors of $\mathbf{A}$ do not vanish.

(ii) The sparseness structure of $\mathbf{A}$ is symmetric, i.e., $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$.

The first assumption allows one to consider diagonal pivoting only; thus, instead of (2), we have

$$\mathbf{PAP}^T = \mathbf{LDU}. \tag{3}$$

Then the second assumption implies that $\mathbf{L}$ and $\mathbf{U}^T$ have the identical sparseness structure. The underlying assumption here is, of course, that numerical cancellation does not occur in the process of triangular factorization.

The symmetric sparseness structure of $\mathbf{A}$ may be characterized by an undirected graph $G = (V, E)$, where V is a finite set of $|V| = n$ elements called *vertices* and

$$E \subset \{(x, y) | \, x, y \in V \text{ and } x \neq y\}$$

is a set of $|E|$ vertex pairs, called *edges*, constructed as follows. Let the vertices be numbered as 1, 2,..., n, then $(i, j) \in E$ if and only if $a_{ij} \neq 0$, where $i \neq j$. Note that the graph G thus obtained represents any matrix which belongs to the equivalence class of matrices $\mathbf{PAP}^T$.

In addition to the two assumptions on $\mathbf{A}$ made above, the following assumption is made for simplicity.

(iii) $\mathbf{A}$ is irreducible, i.e., it cannot be transformed, by symmetric row and column permutations into a block diagonal form.

This implies that the graph G is connected. Thus, the graph G is an undirected, connected graph such that not more than one edge connects the same pair of vertices.

Graph-theoretic terms used in this paper have the same meaning as in [11] unless otherwise defined below.

Given a subset $A \subset V$, the set of vertices *adjacent* to A is denoted by $\mathrm{Adj}(A)$: $\mathrm{Adj}(A) \triangleq \{b \in V - A \mid (a, b) \in E \text{ for some } a \in A\}$. If A consists of a single vertex a, the abbreviation $\mathrm{Adj}(a)$ is used instead of $\mathrm{Adj}(\{a\})$. The set of vertex pairs of A, which do not exist in E, is the *deficiency* of A denoted by $\mathrm{Def}(A)$:

$$\mathrm{Def}(A) \triangleq \{(a, b) \notin E \mid a, b \in A\}.$$

In case of ambiguity, the graph under consideration is referred to: for instance, $\mathrm{Adj}(A)$ with respect to G, $\mathrm{Def}(A)$ w.r.t. G, and so forth. The *section graph* determined by $A \subset V$ is denoted by $G(A)$:

$$G(A) \triangleq (A, E_A),$$

where

$$E_A \triangleq \{(a, b) \in E \mid a, b \in A\}.$$

A set of vertices $A \subset V$ is a *clique* if $\mathrm{Def}(A)$ is void; in other words, A is a clique if A is pairwise adjacent.

For a pair of distinct vertices a, $b \in V$, an a, b *chain* is an ordered set of vertices

$$\{v_1, v_2, \ldots, v_l\}$$

such that $v_1 = a$, $v_l = b$ and $v_{i+1} \in \mathrm{Adj}(v_i)$ $(i = 1, \ldots, l-1)$. A *cycle* of length l is an ordered set of distinct vertices $\{v_1, v_2, \ldots, v_l, v_1\}$ such that $v_{i+1} \in \mathrm{Adj}(v_i)$ $(i = 1, 2, \ldots, l-1)$, and $v_1 \in \mathrm{Adj}(v_l)$. A *separator* is a subset $S \subset V$ such that the section graph $G(V - S)$ consists of two or more *connected components*. An a, b *separator* is *minimal* if none of its proper subsets is an a, b separator.

A graph G is *triangulated* if every cycle of length >3 has a chord, where a chord is an adge connecting two nonconsecutive vertices of the cycle. The following result is fundamental [6, 7, 11, 12].

THEOREM A. *Graph $G = (V, E)$ is triangulated if and only if every minimal a, b separator is a clique.*

Since triangular factorization of matrices is in essence no different from Gaussian elimination [13], one can consider the latter to obtain a corresponding operation on graphs. At the first major step of Gaussian elimination, let a_{11} be the pivot. If $a_{i1} \neq 0$ and $a_{1j} \neq 0$ $(i \neq 1, j \neq 1)$, then after performing the first step, a_{ij} is replaced by $a_{ij} - (a_{i1}a_{1j}/a_{11})$. Under the assumption that no numerical cancellation occurs, this element is still nonzero if $a_{ij} \neq 0$, and will also become nonzero (new fill-in) if $a_{ij} = 0$, that is, when vertices i and j are adjacent to each other.

The foregoing argument leads to the following definition. For a vertex $x \in V$ of graph $G = (V, E)$, the vertex elimination on x is the operation of 1) eliminating x and all its incident edges, and 2) adding edges $\mathrm{Def}(\mathrm{Adj}(x))$ so that $\mathrm{Adj}(x)$ becomes a clique. Thus, Gaussian elimination is interpreted as the successive application of vertex elimination operations. The union of edges added corresponds to the set of fill-ins which are originally zeros in $\mathbf{A}$ and nonzero in $\mathbf{L}$ and $\mathbf{U}$.

For a subset $A \subset V$, the graph obtained from G by eliminating all the vertices of $V - A$ one-by-one is denoted by $G\langle A\rangle$. It is easy to see that $G\langle A\rangle$ is defined independently of ordering by which vertices of $V - A$ are successively eliminated. More specifically, let $C_1 = (X_1, E_1), \ldots$ be connected components of section graph $G(V - A)$. Then $G\langle A\rangle$ can be obtained from G by (i) eliminating all the vertices of $V - A$ and all their incident edges, and (ii) adding edges, if necessary, so that $\mathrm{Adj}(X_i)$ with respect to G becomes a clique for every component C_i. $G\langle A\rangle$ is called the *elimination graph* determined by A.

Let α be a permutation of $\{1, 2, \ldots, n\}$. Then α identifies a row-permutation matrix $\mathbf{P} = \mathbf{P}_\alpha$ in (3). Thus, the triangular factorization of $\mathbf{P}_\alpha \mathbf{A} \mathbf{P}_\alpha^T$ corresponds to the successive application of vertex elimination operations, in which vertices to be eliminated are picked up in the order $\alpha(1), \alpha(2), \ldots, \alpha(n)$; hence α is called an *ordering*. For an ordering α, let $E^{(k)}$ be the set of edges of elimination graph $G\langle\{\alpha(i)\}_{i=k}^n\rangle$ for $k = 1, \ldots, n$. Then the set

$$\mathrm{Trg}(G; \alpha) \triangleq \bigcup_{k=1}^{n} E^{(k)} - E$$

is the set of edges added in the entire course of elimination corresponding to the ordering α. Therefore, the set corresponds to the set of fill-ins created in the triangular factorization of $\mathbf{P}_\alpha \mathbf{A} \mathbf{P}_\alpha^T$. The supergraph of G obtained from G by adding edges of $\mathrm{Trg}(G; \alpha)$ is denoted by $G[\alpha]$:

$$G[\alpha] \triangleq (V, E \cup \mathrm{Trg}(G; \alpha)).$$

Graph $G[\alpha]$ thus represents the sparseness structure of triangular factorization $\mathbf{LDU}$ of $\mathbf{P}_\alpha \mathbf{A} \mathbf{P}_\alpha^T$.

An ordering α is said to be *perfect* if $\mathrm{Trg}(G; \alpha)$ is void. Rose extended Theorem A as follows [6, 11, 12].

THEOREM A′. *For a graph $G = (V, E)$ the following statements are mutually equivalent.*

(i) *There exists a perfect ordering.*

(ii) *G is striangulated.*

(iii) *Every minimal a, b separator is a clique.*

An ordering $\hat{\alpha}$ is said to be optimum (optimal) if no ordering α exists such that $|\mathrm{Trg}(G; \alpha)| < |\mathrm{Trg}(G; \hat{\alpha})|$ ($\mathrm{Trg}(G; \alpha) \Subset \mathrm{Trg}(G; \hat{\alpha})$, where $\Subset$ denotes the strict inclusion). Therefore, an optimum ordering $\hat{\alpha}$ is the best ordering in the sense that it preserves sparseness best.

Now consider a graph $G = (V, E)$ which is not necessarily triangulated. A subset $F \subset \mathrm{Def}(V)$ is called a triangulation of G if $\hat{G} \triangleq (V, E \cup F)$ is triangulated. It then follows from Theorem A′ that $\mathrm{Trg}(G; \alpha)$ is a triangulation. A triangulation $\hat{F}$ is said to be minimum (minimal) if there exists no triangulation F such that $|F| < |\hat{F}|$ ($F \Subset \hat{F}$). Note that a minimum triangulation is necessarily minimal, but the converse is, in general, not true. An example shown in Fig. 1 illustrates this point.

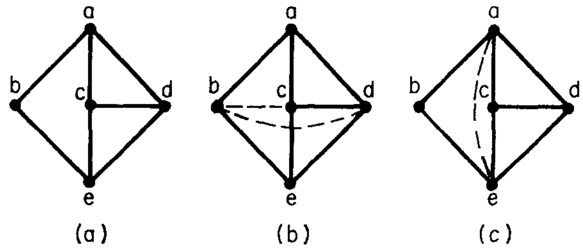

FIG. 1. Minimal triangulation and minimum triangulation. (a) Graph $G = (V, E)$. (b) Minimal triangulation $\{(b, c), (b, d)\}$. (c) Minimum triangulation $\{(a, e)\}$.

3. Minimal Triangulation

For any ordering α of vertices V of a graph $G = (V, E)$ the set $\mathrm{Trg}(G; \alpha)$ is a triangulation as has been shown in the previous section. Then the question arises as to whether there exists an ordering α such that $\mathrm{Trg}(G; \alpha) = F$ for an arbitrarily given triangulation F. A counterexample shown in Fig. 2 gives

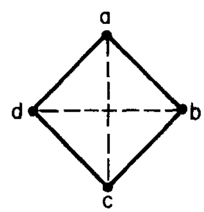

FIG. 2. Nonexistence of ordering α with $\mathrm{Trg}(G; \alpha) = F$, $G = (V, E)$; $V = \{a, b, c, d\}$; $E = \{(a, b), (b, c), (c, d), (d, a)\}$; and $F = \{(a, c), (b, d)\}$.

the answer, where triangulation $F \triangleq \{(a, c), (b, d)\}$ has no corresponding orderings. As far as triangular factorization of matrices is concerned, the class of triangulations that have corresponding orderings is the only one worthwhile considering. With respect to this problem, the following two lemmas give a solution which asserts that the set of minimal (minimum) triangulations cover all the optimal (optimum) orderings and vice versa.

LEMMA 1. *Let F be a minimal (minimum) triangulation of a graph $G = (V, E)$. Then there exists an optimal (optimum) ordering α of V such that* $\mathrm{Trg}(G; \alpha) = F$.

Proof. Since $\hat{G} \triangleq (V, E \cup F)$ is triangulated, there exists a perfect ordering α of V due to Theorem A′. As G is a subgraph of $\hat{G}$, $\mathrm{Trg}(G; \alpha) \subset F$. If $\mathrm{Trg}(G; \alpha) \subsetneq F$, it contradicts the assumption that F is minimal (minimum).

It remains to show that α is optimal (optimum). Suppose α is not optimal (optimum). Then there exists another ordering β such that

$$\mathrm{Trg}(G;\beta) \Subset \mathrm{Trg}(G;\alpha) = F(|\mathrm{Trg}(G;\beta)| < |\mathrm{Trg}(G;\alpha)|),$$

which also contradicts the assumption.

LEMMA 2. *Let α be an optimal (optimum) ordering of the vertices V of a graph $G = (V, E)$. Then* $\mathrm{Trg}(G;\alpha)$ *is a minimal (minimum) triangulation.*

Proof. Suppose $\mathrm{Trg}(G;\alpha)$ is not minimal (minimum), then there exists a triangulation F of G such that $F \Subset \mathrm{Trg}(G;\alpha)(|F| < |\mathrm{Trg}(G;\alpha)|)$. Due to Lemma 1 there exists an ordering β such that $F = \mathrm{Trg}(G;\beta)$. Hence, α is not optimal. By contraposition, we have the lemma.

Summarizing the two lemmas one obtains the following.

THEOREM 1. *A triangulation F of a graph $G = (V, E)$ is minimal (minimum) if and only if there exists an optimal (optimum) ordering α such that* $\mathrm{Trg}(G;\alpha) = F$.

Before presenting a complete characterization of minimal triangulations, certain existing algorithms for finding good orderings are briefly examined. Among them the following two have been widely used [2].

MINIMUM DEGREE ALGORITHM. At each stage of vertex elimination, one chooses a vertex x such that $|\mathrm{Adj}(x)|$ is minimum.

MINIMUM DEFICIENCY ALGORITHM. At each stage of vertex elimination one chooses a vertex x such that $|\mathrm{Def}(\mathrm{adj}(x))|$ is minimum.

These algorithms, in general, fail to give optimal orderings. In a graph shown by the solid line in Fig. 3, it is easy to see that neither of the two algorithms gives an optimal ordering, i.e., a minimal triangulation. In this respect it would be worthwhile to explore the possibility of having an algorithm for finding a *minimal triangulation* F' such that $F' \Subset F$, whenever a given triangulation F is not minimal. Rose [6] made an attempt to find such

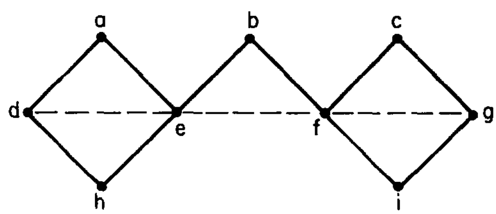

FIG. 3. Counterexample. Initial triangulation $F = \{(d, e), (e, f), (f, g)\}$. Minimal triangulation $F' = \{(d, e), (f, g)\}$.

an algorithm based on, at each stage of vertex elimination, the following information of $G = (X, E)$ and its triangulation F.

(i) $\mathrm{Adj}(x)$ w.r.t. G and $\hat{G}$,

(ii) $\mathrm{Def}(\mathrm{Adj}(x))$ w.r.t. G and $\hat{G}$,

where $\hat{G} \triangleq (X, E^{U}F)$. The fact that the above information is insufficient can be demonstrated by a simple counterexample as shown in Fig. 3. In this example, vertices a, b, and c carry the completely identical information listed above. If vertex b happens to be shosen, the elimination of b necessarily adds edge (e, f). Then $F = \{(d, e), (e, f), (f, g)\}$ can never be reduced to the minimal (minimum) triangulation $F' = \{(d, e), (f, g)\}$.

The following theorem gives a complete characterization of minimal triangulations.

THEOREM 2. *A triangulation F of $G = (V, E)$ is minimal if and only if, for each $(x, y) \in F$, there exists no x, y separator S of G such that S is a clique of the triangulated graph $\hat{G} = (V, E^{\cup}F)$. Its proof depends on a theorem due to Rose* [6, 11, 12].

THEOREM B. *Let $\hat{G} = (V, \hat{E})$ be triangulated with subgraph $G = (V, E)$ where $E \subset \hat{E}$. Then G is triangulated if and only if, for each $(x, y) \in \hat{E} - E$, there exists an x, y separation clique of G.*

Proof of Theorem 2. To prove the "if" part, suppose that there exists a triangulated graph $G' = (V, E^{U}F')$ where $F' \Subset F$. Then it follows from Theorem B that for each $(x, y) \in F - F'$ there exists an x, y separation clique S of G'. Since G' is a supergraph of G and a subgraph of $\hat{G}$, S is an x, y separator of G and also a clique of $\hat{G}$. By contraposition we get the "if" part.

To prove the "only if" part, suppose there exists such an x, y separator S of G. Let $F^* \subset F$ be the set of $(u, v) \in F$ such that S is also a u, v separator of G. Then $G^* = (V, E^{U}(F - F^*))$ is triangulated due to Theorem B because S is a u, v separation clique of G^* for every $(u, v) \in F^*$. Since $F - F^* \Subset F$, triangulation F cannot be minimal, which concludes this part by contraposition.

4. OPTIMAL ORDERING

The characterization of minimal triangulations stated in Theorem 2 leads to a more algorithmic characterization of optimal orderings.

THEOREM 3. *Let $x \in V$ be a vertex of graph $G = (V, E)$. Then there exists an optimal ordering α such that $\alpha(1) = x$ if and only if, for each*

$$(u, v) \in \mathrm{Def}(\mathrm{Adj}(x)),$$

there exists a u, v chain in the section graph

$$G(V - S), \qquad \textit{where} \qquad S \triangleq \{x\} \cup (\mathrm{Adj}(x) - \{u, v\}).$$

Proof. The "if" part: Let β be an optimal ordering of $V - \{x\}$ of the elimination graph $G_x \triangleq G\langle V - \{x\}\rangle$. Define an ordering α of V by

$$\alpha(1) = x, \qquad \alpha(2) = \beta(1),..., \qquad \alpha(n) = \beta(n - 1).$$

Obviously $\mathrm{Trg}(G; \alpha) = \mathrm{Def}(\mathrm{Adj}(x)) \cup \mathrm{Trg}(G_x ; \beta)$ is a triangulation of G. It suffices to prove that $\mathrm{Trg}(G; \alpha)$ is minimal. Suppose that $\mathrm{Trg}(G; \alpha)$ is not minimal. From Theorem 2, for an edge $(y, z) \in \mathrm{Trg}(G; \alpha)$ there exists a y, z separator C of G such that C is a clique of $G[\alpha] = (V, E \cup \mathrm{Trg}(G; \alpha))$. There are two cases to consider.

Case 1. $(y, z) \in \mathrm{Def}(\mathrm{Adj}(x))$ w.r.t. G. Clearly $x \in C$. Since by hypotheses, there exists a y, z chain in $G(X - S')$ with $S' \triangleq \{x\} \cup (\mathrm{Adj}(x) - \{y, z\})$, C must contain at least one vertex $w \in V - \{x\} - \{\mathrm{Adj}(x)\}$. However, edge (x, w) cannot belong to $\mathrm{Trg}(G; \alpha)$ because of the definition of α. Hence C cannot be a clique of $G[\alpha]$. By contraposition, we prove Case 1.

Case 2. $(y, z) \in \mathrm{Trg}(G_x ; \beta)$. Notice that $\mathrm{Trg}(G_x ; \beta)$ is a minimal triangulation of G_x. It can be assumed that C is not an a, b separator of G for any vertex pair $(a, b) \in \mathrm{Def}(\mathrm{Adj}(x))$, for otherwise this case reduces to Case 1. Therefore, $C - \{x\}$ is a y, z separator of G_x and also a clique of $G_x[\beta] = (V - \{x\}, E_x{}^U \mathrm{Trg}(G_x ; \beta))$, where E_x is the set of edges of G_x. Then from Theorem 2, $\mathrm{Trg}(G_x; \beta)$ cannot be a minimal triangulation of G_x, concluding the proof by contraposition.

The "only if" part: Suppose there exists no u, v chain in $G(V - S)$ for some $(u, v) \in \mathrm{Def}(\mathrm{Adj}(x))$, or equivalently S is a u, v separator of G. Since the ordering α has the property that $\mathrm{Def}(\mathrm{Adj}(x)) \subset \mathrm{Trg}(G; \alpha)$, S is a clique of $G[\alpha]$ and $(u, v) \in \mathrm{Trg}(G; \alpha)$. Due to Theorem 2, $\mathrm{Trg}(G; \alpha)$ is not minimal. By contraposition, we have the "only if" part.

From the above proof, the following corollary is immediate.

COROLLARY 1. *Let $x \in V$ be a vertex of a graph $G = (V, E)$ such that there exists an optimal ordering α with $\alpha(1) = x$. Furthermore, let β be an optimal ordering of $V - \{x\}$ of the elimination graph $G_x \triangleq G\langle V - \{x\}\rangle$. Now let an ordering γ of V be defined by*

$$\gamma(1) = \alpha(1) = x, \qquad \gamma(2) = \beta(1),..., \qquad \gamma(n) = \beta(n - 1),$$

then γ is an optimal ordering of graph $G = (V, E)$.

The above-mentioned results, Theorem 3 and Corollary 1, naturally lead to the following algorithm of finding an optimal ordering α. Let $X_1 \subset V$ be the set of vertices of $G = (V, E)$ that satisfies the condition of Theorem 3, and let $v_1 \in X_1$. Next consider the elimination graph $G\langle V_1\rangle$, where $V_1 = V - \{v_1\}$. Similarly $X_2 \subset V$, can be defined as the set of vertices that satisfie the condition with respect to $G\langle V_1\rangle$. Then a vertex $v_2 \in X_2$ is chosen to form the elimination graph $G\langle V_2\rangle$ where $V_2 = V_1 - \{v_2\}$. Repeating this process, an optimal ordering α is obtained with $\alpha(1) = v_1$, $\alpha(2) = v_2, \ldots$. If one wishes, one can list all the optimal orderings without duplication.

The following corollary saves machine time necessary for testing the condition of Theorem 3.

COROLLARY 2. *Let X_1, v_1, and X_2 be as described above, and let W be the set* Adj(v_1) *with respect to $G = (V, E)$. Then $x \in X_2 - W$ if and only if $x \in X_1 - W - \{v_1\}$.*

Proof. To prove the "if" part, let $x \in X_1 - W - \{v_1\}$, and

$$(a, b) \in \mathrm{Def}(\mathrm{Adj}(x))$$

with respect to G. Then from Theorem 3, there exists an a, b chain in $G(V - S)$ where $S \triangleq \{x\} \cup (\mathrm{Adj}(x)$ w.r.t. $G) - \{a, b\}$, i.e., S is not an a, b separator of G. Since $x \notin W \cup \{v_1\}$, it follows that Adj(x) w.r.t. $G\langle V - \{v_1\}\rangle =$ Adj(x) w.r.t. G and Def(Adj(x)) w.r.t. $G\langle V - \{v_1\}\rangle \subset$ Def(Adj(x)) w.r.t. G. Therefore, for $(a, b) \in$ Def(Adj(x)) w.r.t. $G\langle V - \{v_1\}\rangle$ there exists an a, b chain in G that does not pass through vertices of $\{x\} \cup \mathrm{Adj}(x) - \{a, b\}$. If this chain contains v_1, then its subchain $\{e, v_1, f\}$ is simply replaced by $\{e, f\}$ because $e, f \in \mathrm{Adj}(v_1)$. Therefore the modified chain is an a, b chain in $G\langle V - \{v_1\}\rangle$ that does not pass through vertices of $\{x\} \cup \mathrm{Adj}(x) - \{a, b\}$. Hence $x \in X_2 - W$.

To prove the "only if" part, let $x \in X_2 - W$. As is shown above $x \notin W \cup \{v_1\}$ implies that Adj(x) w.r.t. $G\langle V - \{v_1\}\rangle =$ Adj(x) w.r.t. G and Def(Adj(x)) w.r.t. $G\langle V - \{v_1\}\rangle \subset$ Def(Adj(x)) w.r.t. G. Let $(a, b) \in$ Def(Adj(x)) w.r.t. G. There are two cases to separately consider.

Case 1. $(a, b) \notin$ Def(Adj(x)) w.r.t. $G\langle V - \{v_1\}\rangle$. In this case $a, b \in W$ and hence $\{a, v_1, b\}$ is an a, b chain in $G(V - S)$ where $S \triangleq \{x\} \cup \mathrm{Adj}(x) - \{a, b\}$.

Case 2. $(a, b) \in$ Def(Adj(x)) w.r.t. $G\langle V - \{v_1\}\rangle$. In this case there exists an a, b chain in $G\langle V - \{v_1\}\rangle$ that does not pass through vertices of $\{x\} \cup \mathrm{Adj}(x) - \{a, b\}$. If this chain contains subchain $\{e, f\}$ such that $(e, f) \notin E$, then this subchain is replaced by $\{e, v_1, f\}$ in G, because $(e, f) \in$ Def(Adj(v_1)) w.r.t. G. So the modified a, b chain in G does not pass through vertices of

$\{x\}^U$ Adj$(x) - \{a, b\}$, which implies that $x \in X_1 - W - \{v_1\}$. This completes the proof.

An important implication of the corollary is the following: After determining $v_1 \in X_1$, it suffices to test only vertices of Adj(v_1) to obtain X_2. This will significantly reduce the computational time in obtaining an optimal ordering for a large sparse matrix.

Finally, an illustrative example is presented as shown in Fig. 4. Applying Theorem 3, one obtains $X_1 = \{a, b, c, f, g, i\}$. Pick up a vertex from X_1, say a, and perform the vertex elimination to find $G\langle X - \{a\}\rangle$ as shown in Fig. 4(b). Then $X_2 = \{b, c, f, g, i\}$. Pick up a vertex from X_2, say b. Repeating this process, an optimal ordering α is obtained: $\alpha(1) = a$, $\alpha(2) = b$, $\alpha(3) = c$, $\alpha(4) = d$, $\alpha(5) = e$, $\alpha(6) = f$, $\alpha(7) = g$, $\alpha(8) = h$ and $\alpha(9) = i$. The resultant minimal triangulation Trg$(G; \alpha)$ is indicated by the set of edges of broken lines in Fig. 4(c).

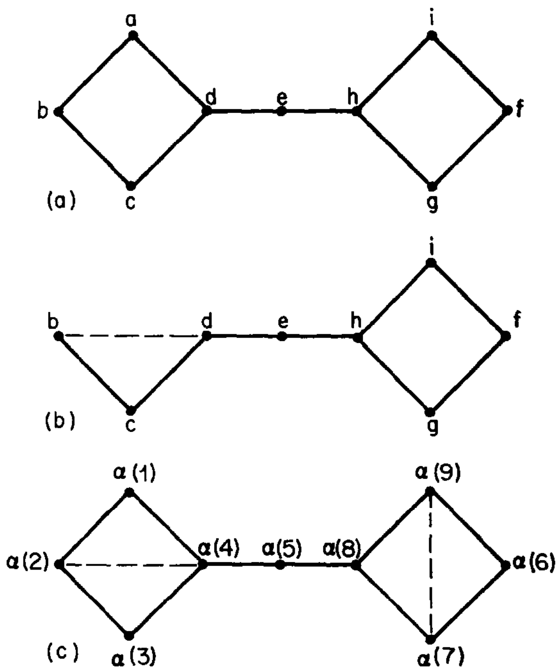

FIG. 4. Illustrative example. (a) $G = (V, E)$. (b) $G\langle V - \{a\}\rangle$. (c) $G[\alpha]$, Trg$(G; \alpha)$ broken lines.

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